

Congruence curves of the Goldstein-Petrich flows

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ABSTRACT. We study the existence of contours which evolve retaining their shapes under the second Goldstein-Petrich flow. We present a proof of the existence, for each integer $n \geq 2$, of a 1-parameter family of non-congruent Goldstein-Petrich contours of \mathbb{R}^2 with symmetry group of order n . Explicit algorithms to compute and visualize the contours and their evolution are given.

1. Introduction

In ref. [GP1], R.E. Goldstein and D.M. Petrich showed that the mKdV equation

$$(1) \quad \kappa_t + \frac{3}{2}\kappa^2\kappa_s + \kappa_{sss} = 0$$

is associated to the flow on the space of unit-speed plane curves $\mathbf{z} : \mathbb{R} \rightarrow \mathbb{C}$ defined by

$$(2) \quad \mathbf{z}_t = -\left(\frac{\kappa^2}{2} + i\kappa_s\right)\mathbf{z}_s, \quad |\mathbf{z}_s| = 1, \quad \kappa = -i\mathbf{z}_{ss}\bar{\mathbf{z}}_s$$

A simple closed curve which evolves retaining its shape under (2) is said to be a GP contour. The existence of GP contours was considered in [GP2, NSW] and examples of closed, non-simple congruence curves of the flow (2) have been examined by Chou and Qu in ref. [CQ]. In [Mu], we exhibited explicit numerical examples of GP contours. Based on these results we wish to prove the following theorem :

Theorem 1. *For every integer $n \geq 2$ there exist $q_n \in (0,1)$ and a 1-parameter family $\{\gamma_{[q,n]}\}_{q \in [0,q_n]}$ of non-congruent GP contours with symmetry group of order*

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n . The evolution of $\gamma_{[q,n]}$ under the second Goldstein-Petrich flow is given by

$$(3) \quad \mathbf{z}_{[q,n]} : (s, t) \in \mathbb{R} \times \mathbb{R} \rightarrow \text{Exp}(t\mu_{[q,n]}) \cdot \gamma_{[q,n]}(s - v_{[q,n]}t) \in \mathbb{R}^2,$$

where $\mu_{[q,n]} \in \mathfrak{e}(2)$ and $v_{[q,n]} \in \mathbb{R}$ are the momentum and the wave velocity of $\gamma_{[q,n]}$. Moreover, there exist a countable set $\mathcal{T}_n \subset [0, q_n)$ such that $\mathbf{z}_{[q,n]}$ is periodic in time, for each $q \in \mathcal{T}_n$.

The material is organized as follows. Section 2 recalls the basic definitions and collects the preliminary results from the existing literature. Section 3 analyzes the explicit integration of GP contours and proves the Theorem. Section 4 develops the numerical algorithms for the construction and the visualization of the 1-parameter families of GP contours with assigned symmetry group.

2. Preliminaries

2.1. Local motions. Denote by $J(\mathbb{R}, \mathbb{R})$ the *total jet space* of smooth \mathbb{R} -valued functions of one independent variable, endowed with its standard coordinates

$$(s, u_{(0)}, u_{(1)}, \dots, u_{(h)}, \dots).$$

If $u : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function, its *prolongation* is defined by

$$j(u) : s \in \mathbb{R} \mapsto \left(s, u|_s, \frac{du}{ds}|_s, \dots, \frac{d^h u}{ds^h}|_s, \dots \right) \in J(\mathbb{R}, \mathbb{R}).$$

A map $\mathfrak{w} : J(\mathbb{R}, \mathbb{R}) \rightarrow \mathbb{R}$ is said a *polynomial differential function* if there exists $w \in \mathbb{R}[x_0, \dots, x_h]$ such that

$$\mathfrak{w}(\mathbf{u}) = w(u_{(0)}, u_{(1)}, \dots, u_{(h)}),$$

for each $\mathbf{u} = (s, u_{(0)}, u_{(1)}, \dots, u_{(h)}, \dots) \in J(\mathbb{R}, \mathbb{R})$. The algebra of polynomial differential functions, $J[\mathbf{u}]$, is endowed with the *total derivative*, defined by

$$D\mathfrak{w} = \sum_{p=0}^{\infty} \frac{\partial w}{\partial u_{(p)}} u_{(p+1)}.$$

A differential function $\mathfrak{w} \in J[\mathbf{u}]$ is a *total divergence* if there exists $\mathfrak{p} \in J[\mathbf{u}]$ such that $\mathfrak{w} = D(\mathfrak{p})$. The primitive \mathfrak{p} is unique up to an additive constant. By $D^{-1}(\mathfrak{w})$ we denote the unique primitive of \mathfrak{w} which vanishes at $\mathbf{u} = \mathbf{0}$. There is another natural differential operator, known as the *Euler operator*, defined by

$$\delta(\mathfrak{w}) = \sum_{\ell=0}^{\infty} (-1)^\ell D^\ell \left(\frac{\partial \mathfrak{w}}{\partial u_{(\ell)}} \right).$$

We now recall three elementary properties :

- $\mathfrak{w} \in J[\mathbf{u}]$ is a total divergence if and only if $\delta(\mathfrak{w}) = 0$;
- for each $\mathfrak{w} \in J[\mathbf{u}]$, $u_{(1)}\delta(\mathfrak{w})$ is a total divergence;
- for each $\mathfrak{w} \in J[\mathbf{u}]$, $u_{(0)}D(\delta(\mathfrak{w}))$ is a total divergence.

We let \mathcal{M} be the space of unit-speed curves $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2 \cong \mathbb{C}$. The arc-length parameter and the curvature are denoted by s and k respectively. Tangent vectors to \mathcal{M} at γ are vector fields $V = (v_1 + iv_2)\gamma'$ along γ satisfying $v'_1 = kv_2$. For each $\mathfrak{v} \in J[\mathbf{u}]$ such that $\delta(u_{(0)}\mathfrak{v}) = 0$, we define a cross section of $T(\mathcal{M})$ by

$$\mathcal{V} : \gamma \in \mathcal{M} \rightarrow (D^{-1}(u_{(0)}\mathfrak{v}) + i\mathfrak{v})|_{j(k)\gamma'} \in T_\gamma(\mathcal{M}).$$

DEFINITION 1. We call \mathcal{V} the *local vector field* associated to $\mathfrak{v} \in J[\mathbf{u}]$. If $\mathfrak{v} = D(\delta(\mathfrak{w}))$, then $u_{(0)}\mathfrak{v}$ is a total divergence and the corresponding local vector field is said to be the *Hamiltonian vector field* with energy \mathfrak{w} . By a *local motion of plane curves* is meant an integral curve of a local vector field.

In other words, a local motion associated to \mathfrak{v} is a smooth map

$$\mathbf{z} = \mathbf{x} + i\mathbf{y} : (s, t) \in \mathbb{R} \times (a, b) \rightarrow \mathbb{C} \cong \mathbb{R}^2$$

such that

$$(4) \quad \mathbf{z}_t = (D^{-1}(u\mathfrak{v})|_{j_s(\kappa)} + i\mathfrak{v}|_{j_s(\kappa)})\mathbf{z}_s, \quad |\mathbf{z}_s| = 1,$$

where

$$(5) \quad \kappa = -i\mathbf{z}_{ss}\bar{\mathbf{z}}_s$$

is the *curvature function*. The *Frenet frame* along \mathbf{z} is the map $\mathcal{A} : \mathbb{R} \times (a, b) \rightarrow \mathbb{E}(2)$ defined by

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 \\ \mathbf{x} & \mathbf{x}' & -\mathbf{y}' \\ \mathbf{y} & \mathbf{y}' & \mathbf{x}' \end{pmatrix}.$$

If we set

$$(6) \quad \mathbf{u} = D^{-1}(u_{(0)}\mathfrak{v}), \quad \mathbf{p} = D\mathfrak{v} + u_{(0)}\mathbf{u},$$

then

$$(7) \quad \Theta := \mathcal{A}^{-1}d\mathcal{A} = \mathcal{K}|_{j_s(\kappa)}ds + \mathcal{P}(\mathfrak{v})|_{j_s(\kappa)}ds,$$

where \mathcal{K} and $\mathcal{P}(\mathfrak{v})$ are the $\mathfrak{e}(2)$ -valued differential functions

$$(8) \quad \mathcal{K} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -u_{(0)} \\ 0 & u_{(0)} & 0 \end{pmatrix}, \quad \mathcal{P}(\mathfrak{v}) = \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{u} & 0 & -\mathbf{p} \\ \mathfrak{v} & \mathbf{p} & 0 \end{pmatrix}.$$

The Maurer-Cartan equation $d\Theta + \Theta \wedge \Theta = 0$ yields

$$(9) \quad \kappa_t = D(D\mathfrak{v} + u_{(0)}D^{-1}(u_{(0)}\mathfrak{v}))|_{j_s(\kappa)}.$$

If the local vector field is Hamiltonian with energy \mathfrak{w} , then (9) takes the form

$$(10) \quad \kappa_t = \mathcal{E}(\delta(\mathfrak{w}))|_{j_s(\kappa)}$$

where

$$\mathcal{E} = (D^3 + D \cdot u_{(0)}D^{-1} \cdot u_{(0)}D)$$

is the canonical Hamiltonian structure of the mKdV hierarchy (cf. chapter 7 of ref. [OI]).

2.2. The Goldstein-Petrich flows and the mKdV hierarchy. According to [GP1] we consider the sequence $\{\mathfrak{v}_n\}_{n \in \mathbb{N}}$ of polynomial differential functions

$$(11) \quad \mathfrak{v}_1 = -u_{(1)} \quad \mathfrak{v}_n = D(D(\mathfrak{v}_{n-1}) + u_{(0)}D^{-1}(u_{(0)}\mathfrak{v}_{n-1})), \quad n \geq 2.$$

Then, the *mKdV hierarchy* is given by

$$(12) \quad u_t = \mathfrak{v}_n|_{j_s(u)}, \quad n \geq 1.$$

Setting

$$\mathfrak{w}_n = \int_0^1 D^{-1}(\mathfrak{v}_{n+1})|_{\epsilon u} u d\epsilon, \quad n \geq 0,$$

we obtain another sequence $\{\mathfrak{w}_n\}_{n \in \mathbb{N}}$ of polynomial differential functions such that

$$(13) \quad \mathfrak{v}_1 = D(\delta(\mathfrak{w}_0)), \quad \mathfrak{v}_n = D(\delta(\mathfrak{w}_{n-1})) = \mathcal{E}(\delta(\mathfrak{w}_{n-2})), \quad n \geq 2.$$

This leads to the bi-Hamiltonian representations of the mKdV hierarchy, namely

$$(14) \quad u_t = D(\delta(\mathfrak{w}_{n-1}))|_{j_s(u)} = \mathcal{E}(\delta(\mathfrak{w}_{n-2}))|_{j_s(u)}, \quad n \geq 2.$$

The first three equations of the mKdV hierarchy are

$$\begin{cases} u_t + u_s = 0 \\ u_t + \frac{3}{2}u^2u_s + u_{sss} = 0, \\ u_t + u_{sssss} + \frac{5}{2}u^2u_{sss} + 10u^2u_su_{ss} + \frac{5}{2}u_s^3 + \frac{15}{8}u^4u_s = 0. \end{cases}$$

DEFINITION 2. The local vector field \mathcal{V}_n associated to \mathfrak{v}_n is called the *n-th flow of Goldstein-Petrich*.

REMARK 1. The Goldstein-Petrich flow \mathcal{V}_n is Hamiltonian with energy \mathfrak{w}_{n-2} , for each $n \geq 2$. Moreover, the curvature function of a local motion of \mathcal{V}_n evolves accordingly to the *n*-th member of the mKdV hierarchy.

3. Goldstein-Petrich contours

3.1. Congruence curves. A unit-speed curve γ which moves without changing its shape under the Goldstein-Petrich flow \mathcal{V}_n is said to be a *congruence curve* of class *n*. From now we consider curves with non-constant curvature. Then, γ is a congruence curve of order *n* if and only if there exist $B : (a, b) \rightarrow \mathbb{E}(2)$ and $v : (a, b) \rightarrow \mathbb{R}$ such that

$$(15) \quad \mathbf{z} : (s, t) \in \mathbb{R} \times (a, b) \rightarrow B(t)\gamma(s - v(t))$$

is a local motion of \mathcal{V}_n .

LEMMA 2. *The function v is linear.*

PROOF. Equation (15) implies that the curvature function of \mathbf{z} is given by

$$\kappa(s, t) = k(s + v(t)),$$

where k is the curvature of γ . From $\kappa_t = \mathfrak{v}_n|_{j_s(\kappa)}$ we find

$$(16) \quad k'|_{s+v(t)} \frac{dv}{dt}|_t = (\mathfrak{v}_n|_{j_s(k)})|_{s+v(t)}.$$

Taking $s_0 \in \mathbb{R}$ such that $k'|_{s_0} \neq 0$ and setting $s = -v(t) + s_0$ in (16) we obtain

$$\frac{dv}{dt}|_t = \frac{v_n|_{j_s(k)}|_{s_0}}{k'|_{s_0}} = \text{constant}.$$

□

As a consequence, we assume that the evolution of a congruence curve is

$$(17) \quad \mathbf{z}(s, t) = B(t) \cdot \gamma(s - vt),$$

where the constant $v \in \mathbb{R}$ is the *wave velocity*. The curvature of a congruence curve of class *n* and wave velocity *v* is a solution of the *stationary mKdV equation*

$$(18) \quad \mathfrak{v}_n|_{j(k)} + vk' = 0.$$

In analogy with (6) we put

$$\mathbf{u}_n = D^{-1}(u_{(0)}\mathfrak{v}_n), \quad \mathbf{p}_n = D(\mathfrak{v}_n) + u_{(0)}\mathbf{u}_n$$

and we consider the $\mathfrak{e}(2)$ -valued polynomial differential function

$$(19) \quad \mathcal{H}(\mathbf{v}_n) = \mathcal{P}(\mathbf{v}_n) + v\mathcal{K},$$

where \mathcal{K} and $\mathcal{P}(\mathbf{v}_n)$ are defined as in (8). An easy inspection shows that k satisfies (18) if and only if

$$(20) \quad (\mathcal{H}(\mathbf{v}_n)|_{j(k)})' = [\mathcal{H}(\mathbf{v}_n), \mathcal{K}]|_{j(k)}.$$

This implies that there exists $\mathbf{m} \in \mathfrak{e}(2)$ such that

$$(21) \quad A \cdot \mathcal{H}(\mathbf{v}_n)|_{j(k)} \cdot A^{-1} = \mathbf{m},$$

where $A : \mathbb{R} \rightarrow \mathbb{E}(2)$ is the Frenet frame along γ . We call \mathbf{m} the *momentum* of γ .

PROPOSITION 3. *Let γ be a congruence curve of class n , with wave velocity $v \in \mathbb{R}$ and momentum \mathbf{m} , then its evolution under \mathcal{V}_n is given by*

$$(22) \quad \mathbf{z}(s, t) = \text{Exp}(t\mathbf{m}) \cdot \gamma(s - vt).$$

PROOF. Let $\mathbf{z}(s, t) = B(t)\gamma(s - vt)$ be the evolution of γ under \mathcal{V}_n . The Frenet frame of \mathbf{z} is

$$(23) \quad \mathcal{A}(s, t) = B(t)A_\gamma(s - vt),$$

where A is the Frenet frame along the curve γ . From (7) we have

$$(24) \quad \mathcal{A}^{-1}d\mathcal{A} = \mathcal{K}|_{j_s(\kappa)}ds + \mathcal{P}(\mathbf{v}_n)|_{j_s(\kappa)}dt.$$

Then, (21), (23) and (24) imply

$$B^{-1}|_t \frac{dB}{dt}|_t = A_\gamma(s - vt) \cdot (\mathcal{H}(\mathbf{v}_n)|_{j(k)})|_{s-vt} \cdot A_\gamma(s + vt)^{-1} = \mathbf{m}.$$

This yields the required result. \square

3.2. Congruence curves of class 2. The curvature of a congruence curve of class two satisfies

$$k''' + \left(\frac{3}{2}k^2 - v\right)k' = 0,$$

where v is the wave velocity. From this we get

$$(k')^2 = -\frac{1}{4}(k^4 + c_2k^2 + c_1k - c_0),$$

where $c_2 = -4v$ and c_1, c_0 are constants of integration. Solutions with $c_1 = 0$ are plane elastic curves. Since closed planar elasticae are not simple [BG], we suppose $c_1 \neq 0$. Eventually scaling γ by a similarity factor, we normalize the curve by $c_1 = 1$ and we assume that the curvature is a periodic solution of

$$(25) \quad (k')^2 = -\frac{1}{4}(k^4 + c_2k^2 + k + c_0).$$

In addition, we require that the polynomial

$$P(t|c_2, c_0) = t^4 + c_2t^2 + t + c_0$$

has two distinct real roots $r_1 > r_2$ and two complex conjugate roots r_3 and r_4 , with $\text{Im}(r_3) > 0$. The coefficients c_2 and c_0 can be written in terms of the parameters $p < 0$ and $q \in (-1, 1)$ by

$$(26) \quad c_{0,p,q} = \frac{(1 + 4p^3q^2)(1 + 4p^3(q^2 - 1))}{16p^4}, \quad c_{2,p,q} = -\frac{1}{2p^2} + p(2q^2 - 1).$$

We set

$$(27) \quad g_{p,q} = -\frac{1}{2p} (1 + p^6 + p^3(4q^2 - 2))^{1/4}, \quad m_{p,q} = \frac{1}{2} + \frac{-1 + p^3(1 - 2q^2)}{2(1 + p^6 + p^3(4q^2 - 2))^{1/2}},$$

and we define

$$\begin{cases} A_{1,p,q} = \frac{1}{2p^2} \sqrt{1 - p^3 + 2q(-p)^{3/2}} (1 - 2q(-p)^{3/2}), \\ A_{2,p,q} = \frac{1}{2p^2} \sqrt{1 - p^3 - 2q(-p)^{3/2}} (1 + 2q(-p)^{3/2}), \\ B_{1,p,q} = \frac{1}{p} \sqrt{1 - p^3 + 2q(-p)^{3/2}}, \\ B_{2,p,q} = \frac{1}{p} \sqrt{1 - p^3 - 2q(-p)^{3/2}}. \end{cases}$$

We denote by $\text{cn}(-|m)$ the Jacobi elliptic cn -function with parameter $m \in (0, 1)$ and we put

$$(28) \quad \begin{cases} \alpha_{1,p,q} = A_{1,p,q} - A_{2,p,q}, & \alpha_{2,p,q} = -(A_{1,p,q} + A_{2,p,q}), \\ \beta_{1,p,q} = B_{1,p,q} - B_{2,p,q}, & \beta_{2,p,q} = -(B_{1,p,q} + B_{2,p,q}). \end{cases}$$

Then,

$$(29) \quad k_{p,q}(s) = \frac{\alpha_{1,p,q} \text{cn}(g_{p,q}s|m_{p,q}) + \alpha_{2,p,q}}{\beta_{1,p,q} \text{cn}(g_{p,q}s|m_{p,q}) + \beta_{2,p,q}}$$

is a periodic solution¹ of (25), with coefficients $c_{0,p,q}$ and $c_{2,p,q}$ and period

$$(30) \quad \omega_{p,q} = \frac{4}{g_{p,q}} \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1 - m_{p,q} \sin^2(\vartheta)}}.$$

For each $p < 0$ and $q \in (-1, 1)$ we let $\gamma_{p,q} : \mathbb{R} \rightarrow \mathbb{R}^2$ be the unit-speed curve with curvature $k_{p,q}$ such that

$$\gamma_{p,q}(0) = (-2p + 4q(-p)^{-1/2}, 0), \quad \gamma'_{p,q}(0) = (0, -1)^t.$$

Since $k_{p,-q}(s) = k_{p,q}(s + \omega_{p,q})$, the curves $\gamma_{p,q}$ and $\gamma_{p,-q}$ are congruent each to the other. If $q = 0$, the curvature is constant and $\gamma_{p,0}$ is a circle with signed radius $2p$. The angular function

$$\theta_{p,q}(s) := \int_0^s k_{p,q}(u) du$$

can be computed in terms of elliptic integrals of the third kind². As a result we obtain

$$(31) \quad \theta_{p,q}(s) = h_{1,p,q}s + h_{2,p,q}\Phi_{2,p,q}(s) + h_{3,p,q}\Phi_{3,p,q}(s),$$

the coefficients $h_{j,p,q}$ and the functions $\Phi_{i,p,q}$ are defined by

$$(32) \quad \begin{cases} h_{1,p,q} = \frac{\alpha_{1,p,q}}{\beta_{1,p,q}}, \\ h_{2,p,q} = \frac{\alpha_{2,p,q}\beta_{1,p,q} - \alpha_{1,p,q}\beta_{2,p,q}}{g_{p,q}\sqrt{(\beta_{2,p,q} - \beta_{1,p,q})(\beta_{1,p,q} + \beta_{2,p,q})(\beta_{1,p,q}^2(1 - m_{p,q}) - \beta_{2,p,q}^2 m_{p,q})}}, \\ h_{3,p,q} = -\frac{\alpha_{2,p,q}\beta_{1,p,q} - \alpha_{1,p,q}\beta_{2,p,q}}{g_{p,q}\beta_{1,p,q}\beta_{2,p,q}\sqrt{1 - m_{p,q}}} \end{cases}$$

¹See ref. [BF], pg. 133

²See ref. [La], pg. 67-69.

and by

$$(33) \quad \begin{cases} \Phi_{2,p,q}(s) = \operatorname{arctanh} \left(\sqrt{\frac{(1-m_{p,q})\beta_{1,p,q}^2 + m_{p,q}\beta_{2,p,q}}{(\beta_{1,p,q}^2 - \beta_{2,p,q}^2)(\beta_{1,p,q} + \beta_{2,p,q})}} \operatorname{sd}(g_{p,q}s|m_{p,q}) \right), \\ \Phi_{3,p,q}(s) = \Pi \left(\frac{\beta_{1,p,q}^2}{\beta_{2,p,q}^2}, \frac{1}{2}(\pi - 2\operatorname{am}(g_{p,q}s|m_{p,q}), \frac{m_{p,q}}{1-m_{p,q}}) \right) - \Pi \left(\frac{\beta_{1,p,q}^2}{\beta_{2,p,q}^2}, \frac{\pi}{2}, \frac{m_{p,q}}{1-m_{p,q}} \right), \end{cases}$$

where

$$\begin{cases} \Pi(n, \phi, m) = \int_0^\phi \frac{d\theta}{(1-n\sin^2(\theta))\sqrt{1-m\sin^2(\theta)}}, \\ \operatorname{am}(s, m) = \int_0^s \operatorname{dn}(u|m)du \end{cases}$$

are the integral of the third kind and the Jacobi amplitude respectively.

PROPOSITION 4. *The curve $\gamma_{p,q}$ is given by*

$$(34) \quad \gamma_{p,q} = 2e^{i\theta_{p,q}} ((2k_{p,q}^2 + c_{2,p,q}) + 4i\kappa'_{p,q}).$$

PROOF. We set

$$(35) \quad \eta_{1,p,q} = -\frac{1}{2}k_{p,q} - \frac{1}{4}c_{2,p,q}, \quad \eta_{2,p,q} = -\kappa'_{p,q}.$$

Then,

$$(36) \quad \mathcal{H}(\mathbf{v}_2)|_{j(k_{p,q})} = \begin{pmatrix} 0 & 0 & 0 \\ \eta_{1,p,q} & 0 & -1/8 \\ \eta_{2,p,q} & 1/8 & 0 \end{pmatrix}.$$

The Frenet frame field of a unit-speed curve γ with curvature $k_{p,q}$ and initial condition $\gamma'(0) = (1, 0)^t$ is

$$(37) \quad A = \begin{pmatrix} 1 & 0 & 0 \\ \gamma_1 & \cos(\theta_{p,q}) & -\sin(\theta_{p,q}) \\ \gamma_2 & \sin(\theta_{p,q}) & \cos(\theta_{p,q}) \end{pmatrix}.$$

Denote by

$$\mathbf{m} = \begin{pmatrix} 0 & 0 & 0 \\ m_1 & 0 & -m_3 \\ m_2 & m_3 & 0 \end{pmatrix}$$

the momentum of γ . From (21) we have

$$(38) \quad A^{-1} \cdot \mathcal{H}(\mathbf{v}_2)|_{j(k_{p,q})} \cdot A = \mathbf{m}.$$

Combining (36), (37) and (38) we obtain

$$\gamma = 8i(e^{i\theta_{p,q}}(\eta_{1,p,q} + i\eta_{2,p,q}) + (m_1 + im_2)).$$

Then,

$$\tilde{\gamma} = 8e^{i\theta_{p,q}}(\eta_{1,p,q} + i\eta_{2,p,q})$$

is a unit-speed curve with curvature $k_{p,q}$ and initial conditions

$$\tilde{\gamma}(0) = (-2p + 4q(-p)^{-1/2}, 0)^t, \quad \tilde{\gamma}'(0) = (0, -1)^t.$$

This implies the required result. \square

Since $\eta_{1,p,q} + i\eta_{2,p,q}$ is periodic, with period $\omega_{p,q}$, we deduce :

COROLLARY 5. *The curve $\gamma_{p,q}$ is closed if and only if*

$$(39) \quad \Lambda_{p,q} = \frac{1}{2\pi} \int_0^{\omega_{p,q}} k_{p,q}(u) du = \frac{\ell}{n} \in \mathbb{Q},$$

where $\ell, n \in \mathbb{Z}$ are relatively prime integers, with $\ell \geq 0$.

REMARK 6. The integer ℓ is the turning number, $|n|$ is the order of the symmetry group. In particular, for a simple curve the integer ℓ is 1. If $q \neq 0$, the elliptic curve parameterized by $k_{p,q}$ and $k'_{p,q}$ intersects the Ox -axis in two points. Then, the four vertex theorem implies $|n| > 1$.

3.3. Proof of Theorem 1. We fix a positive integer $n > 1$. We define the *characteristic curve*

$$\Sigma_n = \{(p, q) \in \mathbb{R}^{-1} \times (-1, 1) : \Lambda_{p,q} = -1/n\},$$

and we let Σ_n^+ be the set of all $(p, q) \in \Sigma_n$ such that $q \geq 0$. Since the function

$$\Lambda : (p, q) \in \mathbb{R}^- \times (-1, 1) \rightarrow \Lambda_{p,q} \in \mathbb{R}$$

satisfies

$$(40) \quad \Lambda_{p,q} = \Lambda_{p,-q}, \quad \Lambda_{p,0} = -\frac{1}{\sqrt{1-p^3}}, \quad \partial_p \Lambda|_{p,0} = -\frac{3p^2}{2(1-p^3)^{3/2}} < 0,$$

then there exist a maximal $\epsilon_n \in (0, 1]$ and a unique real-analytic even function

$$(41) \quad \phi_n : (-\epsilon_n, \epsilon_n) \rightarrow \mathbb{R}^-$$

such that

$$\phi_n(0) = (1 - n^2)^{1/3}, \quad (\phi_n(q), q) \in \Sigma_n, \quad \forall q \in (-\epsilon_n, \epsilon_n).$$

We define

$$(42) \quad \gamma_{[q,n]} := \gamma_{\phi_n(q), q},$$

and we consider the one-parameter family $\{\gamma_{[q,n]}\}_{q \in (-\epsilon_n, \epsilon_n)}$ of closed curves with curvature functions $k_{[q,n]} = k_{\phi_n(q), q}$. We let $\omega_{[q,n]}$ be the period of $k_{[q,n]}$. Then,

$$\mathfrak{K}_{[n]} : (s, q) \in \mathbb{R} \times (-\epsilon_n, \epsilon_n) \rightarrow k_{[q,n]}(s) \in \mathbb{R}$$

is a real-analytic function, periodic in s , satisfying

$$\mathfrak{K}_{[n]}(s, 0) = (2\phi_n(q))^{-1} < 0.$$

It follows that there exists $\epsilon'_n \in (0, \epsilon_n]$ such that $\gamma_{[q,n]}$ is strictly convex and satisfies

$$\frac{1}{2\pi} \int_0^{\omega_{[q,n]}} k_{[q,n]}(u) du = -\frac{1}{n},$$

for each $q \in (-\epsilon'_n, \epsilon'_n)$. This implies (cf. [MN]) that $\gamma_{[q,n]}$ is a simple curve, for every $q \in (-\epsilon'_n, \epsilon'_n)$. We set

$$q_n = \sup\{q \in (0, \epsilon_n) : \gamma_{[\tilde{q},n]} \text{ is a simple curve, } \forall \tilde{q} \in (0, q)\}.$$

Then, $\{\gamma_{[q,n]}\}_{q \in [0, q_n]}$ is a one-parameter family of simple congruence curves of class 2, with symmetry group of order n . Since the curves of the family have different

lengths, they are not congruent each to the other. The momentum $\mathbf{m}_{[q,n]}$ and the wave velocity $v_{[q,n]}$ of $\gamma_{[q,n]}$ are given by

$$(43) \quad \mathbf{m}_{[q,n]} = \frac{1}{8} \begin{pmatrix} 0 & 0 & 0 \\ \mu_{[q,n]} & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mu_{[q,n]} = -2\phi_n(q) + \frac{4q}{\sqrt{-\phi_n(q)}}.$$

and by

$$(44) \quad v_{[q,n]} = \frac{1}{4} \left(\frac{1}{2\phi_n(q)^2} - \phi_n(q)(-1 + 2q^2) \right).$$

From (22), (43) and (44) we see that the evolution of $\gamma_{[q,n]}$ is

$$(45) \quad \mathbf{z}_{[q,n]}(s, t) = e^{it/8} \gamma_{[q,n]}(s - v_{[q,n]}t) + \mu_{[q,n]} \rho(t),$$

where

$$\rho(t) = \sin(t/8) + i(1 - \cos(t/8)).$$

Therefore, $\mathbf{z}_{[q,n]}(s, t)$ is periodic in time if and only if

$$\frac{4\pi v_{[q,n]}}{n\omega_{[q,n]}} \in \mathbb{Q}.$$

Since the function

$$(46) \quad T_n : q \in [0, q_n) \rightarrow \frac{4\pi v_{[q,n]}}{n\omega_{[q,n]}}$$

is non-constant and real-analytic, then there exists a countable set $\mathcal{T}_n \subset [0, q_n)$ such that the evolution of $\gamma_{[q,n]}$ is periodic, for all $q \in \mathcal{T}_n$.

3.4. Example. Consider the family $\{\gamma_{[q,7]}\}_{q \in [0,1)}$. The function ϕ_7 is defined for all $q \in (-1, 1)$ and the upper part Σ_7^+ of the characteristic curve is parameterized $q \in [0, 1) \rightarrow (\phi_7(q), q) \in \Sigma_7^+$. The approximate value of q_7 is 0.8013658294677735. The behavior of the family is illustrated in the Figures 1, 2, 3 and 4.

REMARK 7. Numerical experiments show that the characteristic curve Σ_n is always the graph of the function $\phi_n : (-1, 1) \rightarrow \mathbb{R}$. Furthermore, there exists a well defined *separating value* $q_n \in (0, 1)$ such that $\gamma_{[q,n]}$ is simple if and only if $|q| < q_n$. The experimental evidence also suggest that each GP-contour is equivalent, up to a similarity of \mathbb{R}^2 , to a curve of the form $\gamma_{[q,n]}$, with $n \geq 2$ and $q \in (0, q_n)$.

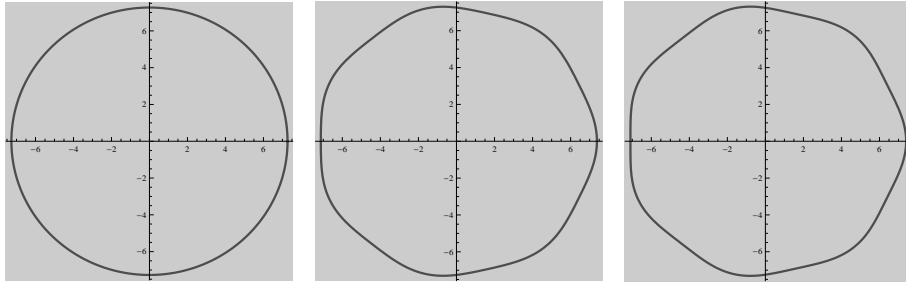
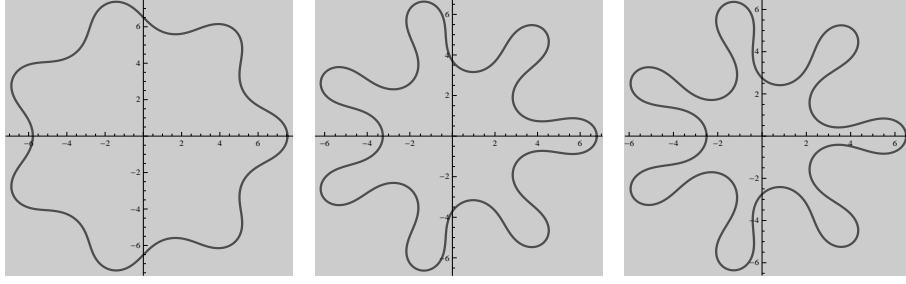


FIGURE 1. The curves $\gamma_{[0,7]}$, $\gamma_{[0.06,7]}$ and $\gamma_{[q_1,7]}$.

FIGURE 2. The curves $\gamma_{[0.4,7]}$, $\gamma_{[q_2,7]}$ and $\gamma_{[0.75,7]}$.

The numerical value of q_n can be found by the mean of the following procedure (see also Step 6 of Section 4) :

- compute q'_n such that $c_2(\phi_n(q'_n), q'_n) = 0$;
- compute $q''_n \in (q'_1, 1)$ such that

$$\eta_{1, \phi_n(q''_n), q''_n} \left(\frac{\omega_{[q''_n, n]}}{2} \right) = 0,$$

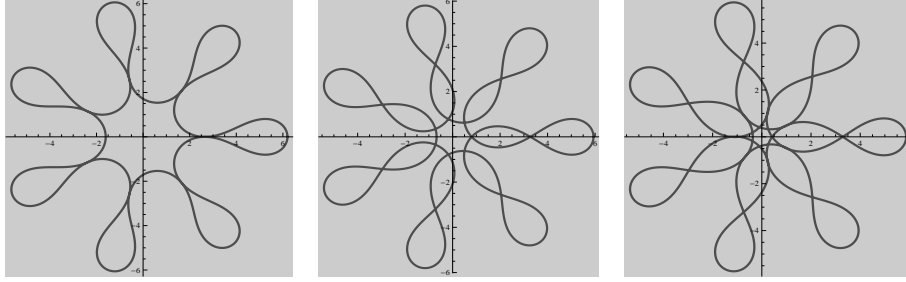
where $\eta_{1,p,q}$ is defined as in (35).

- for each $q \in (q'_1, q''_1]$ compute $s_{[q,n]} \in [0, \frac{\omega_{[q''_n, n]}}{2}]$ such that

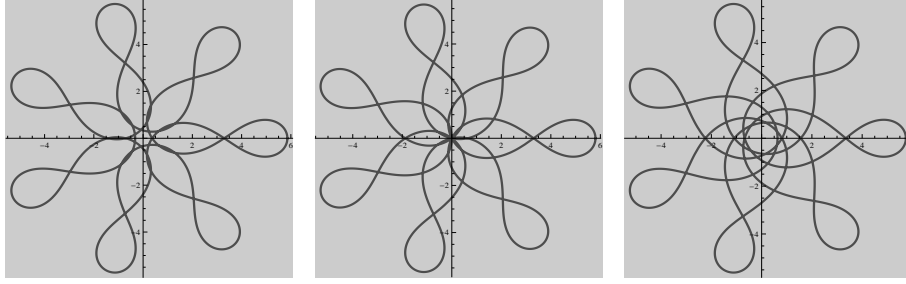
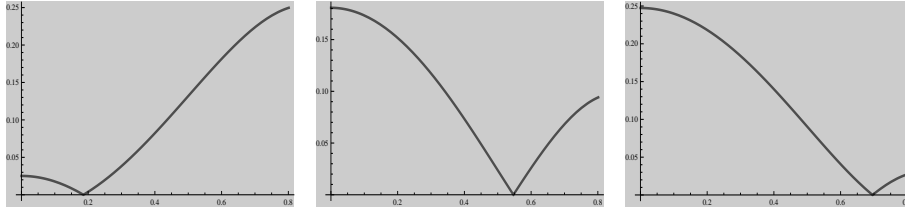
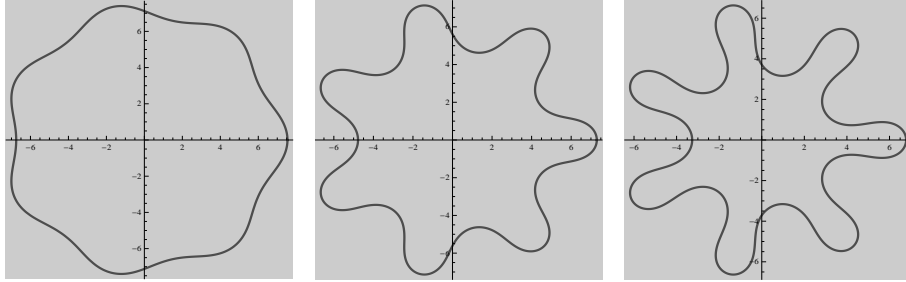
$$g_{\phi_n(q), q} \text{cn}(s_{[q,n]} | m_{\phi_n(q), q}) = - \frac{\alpha_{2, \phi_n(q), q} - \beta_{2, \phi_n(q), q} \sqrt{-c_{2, \phi_n(q), q}}}{\alpha_{1, \phi_n(q), q} - \beta_{1, \phi_n(q), q} \sqrt{-c_{2, \phi_n(q), q}}};$$

- q_n is the unique zero of the function

$$q \in (q'_1, q''_1] \rightarrow |\gamma_{[q,n]}(s_{[q,n]}) - \gamma_{[q,n]}(-s_{[q,n]})|.$$

FIGURE 3. The curves $\gamma_{[q^*,7]}$, $\gamma_{[0.845,7]}$ and $\gamma_{[q_3,7]}$.

REMARK 8. Congruence curves which evolve periodically in time can be computed as follows: consider the function T_n and set $I_n = \text{Im}(T_n)$. For each $\ell/h \in I_n \cap \mathbb{Q}$ there exists a unique $\hat{q}_n(\ell, h) \in [0, q_n)$ such that $T_n(\hat{q}_n(\ell, h)) = \ell/h$. The explicit evaluation of $\hat{q}_n(\ell, h)$ can be made via numerical routines (see Steps 9 and 10 of Section 4). Thus, the family of simple, closed congruence curves with symmetry group of order n and periodic evolution in time is $\{\gamma_{[\hat{q}_n(u), n]}\}_{u \in I_n \cap \mathbb{Q}}$.

FIGURE 4. The curves $\gamma_{[0.86,7]}$, $\gamma_{[q_4,7]}$ and $\gamma_{[0.9,7]}$.FIGURE 5. The functions $T_7(q) - \ell/h$, with $\ell/h = -2/9, -1/15, 0$.FIGURE 6. The curves $\gamma_{[\hat{q}_7(-2/9),7]}$, $\gamma_{[\hat{q}_7(-1/15),7]}$ and $\gamma_{[\hat{q}_7(0),7]}$.

4. Numerical computations and visualization

In this section we show how to translate the results and the computations of Section 2 into numerical and graphical routines implemented the software *Mathematica 7.0*.

- **Step 1.** Define the coefficients $c_{0,p,q}$, $c_{2,p,q}$, $\alpha_{j,p,q}$, $\beta_{j,p,q}$, $j = 1, 2$, $g_{p,q}$ and $m_{p,q}$ as in (26), (28) and (27) :

$$\begin{aligned} C0[p-, q-] &:= \frac{(1+4p^3q^2)(1+4p^3(-1+q^2))}{16p^4}; \\ C2[p-, q-] &:= -\frac{1}{2p^2} + p(-1+2q^2); \end{aligned}$$

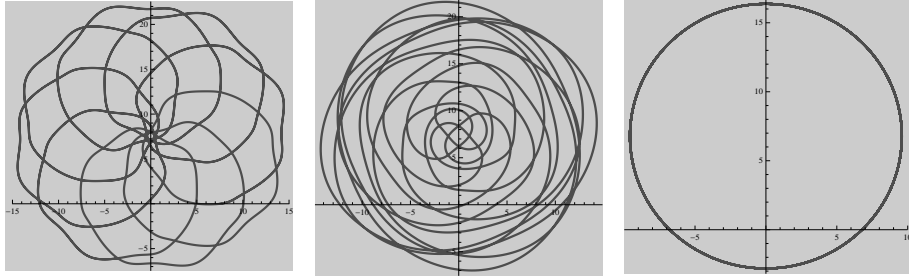


FIGURE 7. Trajectories of the points $\gamma_{[\hat{q}_7(-\frac{2}{9}), 7]}(0)$, $\gamma_{[\hat{q}_7(-\frac{2}{15}), 7]}(0)$ and $\gamma_{[\hat{q}_7(0), 7]}(0)$.

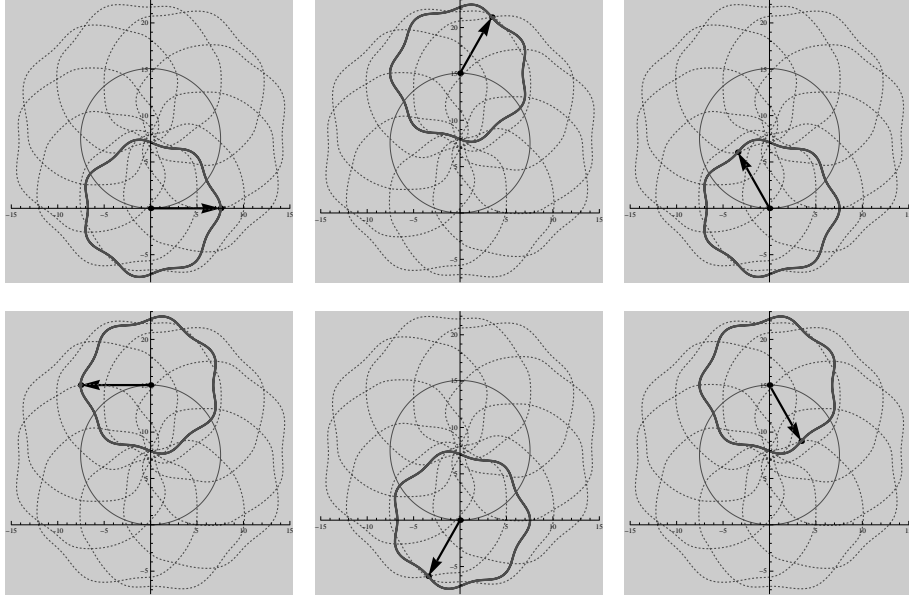


FIGURE 8. Evolution of the curve $\gamma_{[\hat{q}_7(-\frac{2}{9}), 7]}$.

$$\begin{aligned}
 \alpha 1[p-, q-] &:= \frac{\sqrt{\frac{1}{p^2} - p - \frac{2q}{\sqrt{-p}}}(1+2(-p)^{3/2}q)}{2p} + \frac{\sqrt{1-p^3+2(-p)^{3/2}q}(1+2\sqrt{-ppq})}{2p^2}; \\
 \alpha 2[p-, q-] &:= \frac{\sqrt{\frac{1}{p^2} - p - \frac{2q}{\sqrt{-p}}}(1+2(-p)^{3/2}q)}{2p} - \frac{\sqrt{1-p^3+2(-p)^{3/2}q}(1+2\sqrt{-ppq})}{2p^2}; \\
 \beta 1[p-, q-] &:= \sqrt{\frac{1}{p^2} - p - \frac{2q}{\sqrt{-p}}} + \frac{\sqrt{1-p^3+2(-p)^{3/2}q}}{p}; \\
 \beta 2[p-, q-] &:= \sqrt{\frac{1}{p^2} - p - \frac{2q}{\sqrt{-p}}} - \frac{\sqrt{1-p^3+2(-p)^{3/2}q}}{p}; \\
 m[p-, q-] &:= \frac{-1+p^3(1-2q^2)+\sqrt{1+p^6+p^3(-2+4q^2)}}{2\sqrt{1+p^6+p^3(-2+4q^2)}}; \\
 g[p-, q-] &:= -\frac{(1+p^6+p^3(-2+4q^2))^{1/4}}{2*p};
 \end{aligned}$$

- **Step 2.** Define the curvature $k_{p,q}$ and its period $\omega_{p,q}$ (cf. (29) and (30)) :

$$\begin{aligned} k[s-, p-, q-] &:= \frac{\alpha 1[p, q] * \text{JacobiCN}[g[p, q] * s, m[p, q]] + \alpha 2[p, q]}{\beta 1[p, q] * \text{JacobiCN}[g[p, q] * s, m[p, q]] + \beta 2[p, q]}; \\ \text{Dk}[s-, p-, q-] &:= \text{Evaluate}[D[k[s, p, q], s]]; \\ \omega[p-, q-] &:= \frac{4}{g[p, q]} \text{EllipticK}[m[p, q]]; \end{aligned}$$

- **Step 3.** Compute the angular function $\theta_{p,q}$ (cf. (31),(32) and (33)) :

$$\begin{aligned} h1[p-, q-] &:= \frac{\alpha 1[p, q]}{\beta 1[p, q]}; \\ h2[p-, q-] &:= \frac{(\alpha 2[p, q] \beta 1[p, q] - \alpha 1[p, q] \beta 2[p, q])}{g[p, q] * \sqrt{(-\beta 1[p, q] + \beta 2[p, q])(\beta 1[p, q] + \beta 2[p, q])((-1 + m[p, q]) \beta 1[p, q]^2 - m[p, q] \beta 2[p, q]^2)}}; \\ h3[p-, q-] &:= -\sqrt{\frac{1}{1 - m[p, q]}} * \frac{(\alpha 2[p, q] \beta 1[p, q] - \alpha 1[p, q] \beta 2[p, q])}{g[p, q] \beta 1[p, q] \beta 2[p, q]}; \\ \Phi 2[s-, p-, q-] &:= \text{ArcTanh} \left[\frac{\text{JacobiSD}[sg[p, q], m[p, q]]}{\sqrt{\frac{(\beta 1[p, q] - \beta 2[p, q])(\beta 1[p, q] + \beta 2[p, q])}{-(-1 + m[p, q]) \beta 1[p, q]^2 + m[p, q] \beta 2[p, q]^2}}} \right]; \\ \Phi 3[s-, p-, q-] &:= -\text{EllipticPi} \left[\frac{\beta 1[p, q]^2}{\beta 2[p, q]^2}, \frac{m[p, q]}{-1 + m[p, q]} \right] + \\ &\text{EllipticPi} \left[\frac{\beta 1[p, q]^2}{\beta 2[p, q]^2}, \frac{1}{2}(\pi - 2 \text{JacobiAmplitude}[sg[p, q], m[p, q]]), \frac{m[p, q]}{-1 + m[p, q]} \right]; \\ \theta[s-, p-, q-] &:= h1[p, q] * s + h2[p, q] * \Phi 2[s, p, q] + h3[p, q] * \Phi 3[s, p, q]; \end{aligned}$$

- **Step 4.** Compute the function ϕ_n and the one-parameter family of closed curves $\{\gamma_{[q,n]}\}_{q \in [0,1]}$ (cf. (35),(34),(41) and (42)) :

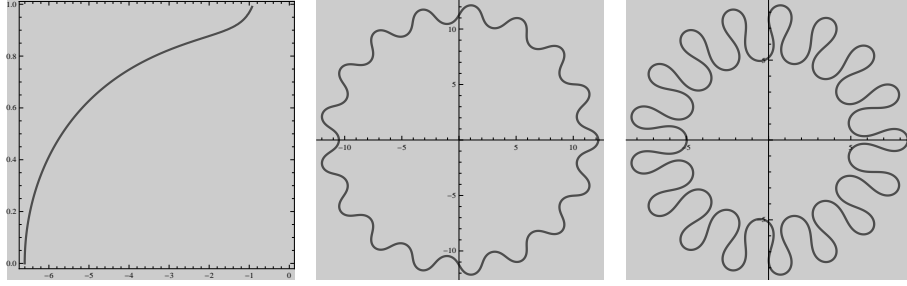
$$\begin{aligned} \Lambda[p-, q-] &:= \frac{1}{2\beta 1} \theta[\omega[p, q], p, q]; \\ \eta[s-, p-, q-] &:= \left\{ \frac{C2[p, q]}{4} + \frac{k[s, p, q]^2}{2}, \text{Dk}[s, p, q] \right\}; \\ R[s-, p-, q-] &:= \{\{\text{Cos}[\theta[s, p, q]], -\text{Sin}[\theta[s, p, q]]\}, \{\text{Sin}[\theta[s, p, q]], \text{Cos}[\theta[s, p, q]]\}\}; \\ \varphi[q-, n-] &:= \text{Evaluate}[\text{FindRoot}[\Lambda[p, q] + \frac{1}{n} == 0, \{p, -\sqrt[3]{-1 + n^2}, -0.1\}], \text{Method} \rightarrow \text{"Brent"}][[1]][[2]]; \\ \gamma[s-, q-, n-] &:= 8 * R[s, \varphi[q, n], q].\eta[s, \varphi[q, n], q]; \end{aligned}$$

- **Step 5.** Visualize Σ_n and the curve $\gamma_{[q,n]}$:

$$\begin{aligned} \Sigma[n-] &:= \text{ContourPlot}[\Lambda[p, q] == -1/n, \{p, -\sqrt[3]{-1 + n^2}, -0.01\}, \{q, 0.00001, 0.99\}, \\ &\text{ContourStyle} \rightarrow \{\text{GrayLevel}[0.3], \text{Thickness}[0.008]\}, \text{Background} \rightarrow \text{GrayLevel}[0.8], \\ &\text{PlotPoints} \rightarrow 50]; \\ \text{CURVE}[q-, n-] &:= \text{ParametricPlot}[\text{Evaluate}[\gamma[s, q, n], \{s, 0, n * \omega[\varphi[q, n], q]\}, \\ &\text{PlotStyle} \rightarrow \{\text{GrayLevel}[0.3], \text{Thickness}[0.008]\}, \text{Background} \rightarrow \text{GrayLevel}[0.8], \\ &\text{PlotRange} \rightarrow \text{All}]; \end{aligned}$$

- **Step 6.** Specify the order of the symmetry group and compute q_n :

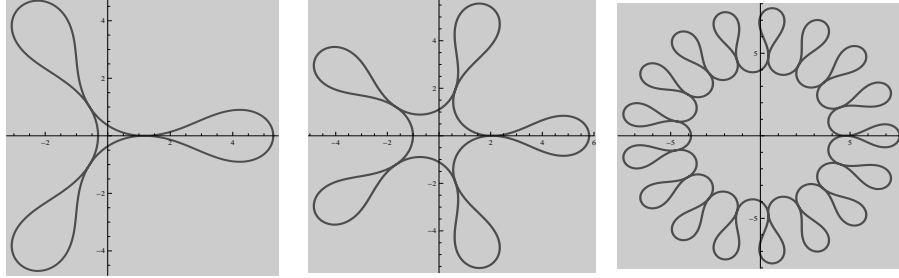
$$\begin{aligned} nn &:= 5 \\ f1[q-] &:= \text{Abs}[C2[\varphi[q, nn], q]]; \\ F1 &:= \text{Plot}[f1[q], \{q, 0, 0.99\}]; \\ f2[q-] &:= \text{Abs} \left[\frac{C2[\varphi[q, nn], q]}{4} + \frac{k[\frac{1}{2} * \omega[\varphi[q, nn], q], \varphi[q, nn], q]^2}{2} \right]; \\ F2 &:= \text{Plot}[f2[q], \{q, 0, 0.99\}]; \\ Q1 &:= \text{Evaluate}[\text{First}[\text{Sort}[\text{InputForm}[F1][[1, 1]][[1, 3]][[2]][[1]], \#1[[2]] < \#2[[2]] \&]]]; \\ Q2 &:= \text{Evaluate}[\text{First}[\text{Sort}[\text{InputForm}[F2][[1, 1]][[1, 3]][[2]][[1]], \#1[[2]] < \#2[[2]] \&]]]; \\ s1[q-] &:= \end{aligned}$$

FIGURE 9. Σ_{17} and the curves $\gamma_{[0.5,17]}$, $\gamma_{[0.8,17]}$.

```

1/g[φ[q,nn],q]InverseJacobiCN [ - (α2[φ[q,nn],q]-β2[φ[q,nn],q]√-C2[φ[q,nn],q]/2) /
α1[φ[q,nn],q]-β1[φ[q,nn],q]√-C2[φ[q,nn],q]/2 ,
m[φ[q,nn],q]];
f3[q_-]:=Norm[γ[s1[q],q,nn]-γ[-s1[q],q,nn]];
steps:=7; initialpoint:=1/2(Q1[[1]]+Q2[[1]]); internalparameter[1]:=1/30;
internalparameter[2]:=20;
QQ[y_-,δ_-,k_-]:=First[Sort[Table[{f3[q],q},{q,y-δ,y+δ,1/k}]]];
S[1,y_-,δ_-,k_-]:=QQ[y,δ,k];
S[m_-,y_-,δ_-,k_-]:=S[m-1,S[m-1,y,δ,k][[2]],δ/(2^(m-1)),k*(2^(m-1))];
Qn:=Evaluate[S[steps,initialpoint,internalparameter[1],internalparameter[2]]];

```

FIGURE 10. The "separating" curves $\gamma_{[q_3,3]}$, $\gamma_{[q_5,5]}$ and $\gamma_{[q_{17},17]}$.

- **Step 7.** Compute the evolution of the congruence curves :

```

μ1[q_-,n_-]:=-2*φ[q,n]+(4*q)/√-φ[q,n];
μ[q_-,n_-]:=1/8({0,0,0},{μ1[q,n],0,-1},{0,1,0});
v[q_-,n_-]:=1/4((1/(2*φ[q,n]^2)-φ[q,n](2q^2-1)));
z[s_-,t_-,q_-,n_-]:={ {Cos[t/8],-Sin[t/8]}, {Sin[t/8],Cos[t/8]} }.γ[s-v[q,n]*t,q,n]+
μ1[q,n]{Sin[t/8],-Cos[t/8]+1};

```

- **Step 9.** Compute the function T_n :

```

T[q_-,n_-]:=4*(Pi*v[q,n])/(n*ω[φ[q,n],q]);

```

• **Step 10.** Specify the order of the symmetry group, take $u/w \in I_7 \cap \mathbb{Q}$ and compute $\hat{q}_n(u, w)$:

```

nnn:=7; u:=-2; w:=9;
T1:=Plot [Abs [T[q, nnn] -  $\frac{u}{w}$ ] , {q, 0, 0.99}] ;
QPA:=Evaluate[First[Sort[InputForm[T1][[1, 1]][[1, 3]][[2]][[1]], #1[[2]] < #2[[2]]&]];
steps:=6
initialpoint:=QPA[[1]];
internalparameter[1]:=1/30;
internalparameter[2]:=20;
QP1[y-,  $\delta$ -, k-]:=First [Sort [Table [{ Abs [T[q, nnn] -  $\frac{u}{w}$ ] , q} , {q, y -  $\delta$ , y +  $\delta$ , 1/k}]]] ;
SQP1[1, y-,  $\delta$ -, k-]:=QP1[y,  $\delta$ , k];
SQP1[m-, y-,  $\delta$ -, k-]:=SQP1 [m - 1, SQP1[m - 1, y,  $\delta$ , k][[2]],  $\delta / (2^{m-1})$  , k * (2m-1) ] ;
QPA2:=Evaluate[SQP1[steps, initialpoint, internalparameter[1], internalparameter[2]]];
QP:=QPA2[[2]];

```

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